Fuzzy geometry of phase space and quantization of massive fields

2008 J. Phys. A: Math. Theor. 41 164071

(http://iopscience.iop.org/1751-8121/41/16/164071)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 164071 (10pp) doi:10.1088/1751-8113/41/16/164071

Fuzzy geometry of phase space and quantization of massive fields

S N Mayburov

Lebedev Institute of Physics, Leninsky Prospect 53, Moscow 117924, Russia

E-mail: mayburov@sci.lebedev.ru

Received 30 October 2007, in final form 31 January 2008
Published 9 April 2008
Online at stacks.iop.org/JPhysA/41/164071

Abstract

The quantum spacetime and the phase space with fuzzy structure are investigated as the possible quantization formalism. In this theory, the state of the nonrelativistic particle m corresponds to the element of fuzzy ordered set (Foset), i.e. the fuzzy point. Due to Foset partial (weak) ordering, the m space coordinate x acquires principal uncertainty σ_x. It is shown that Schrödinger formalism of quantum mechanics can be completely derived from consideration of m evolution in fuzzy phase space with minimal number of axioms.

PACS numbers: 03.65.Ca, 02.10.Ab, 02.40.−k, 11.10.Cd

1. Introduction

Quantum spacetime and its relation to axiomatic of quantum mechanics (QM) and field theory is now actively discussed from the different angles [1–3]. In particular, it was proposed that the fundamental properties of spacetime metrics and topology can be modified significantly at the Planck scale [4–6]. Our work is largely motivated by these ideas, which will be studied in the framework of sets theory, exploring the various set structures of spacetime manifold M_{ST}. For example, in one-dimensional Euclidean geometry, the elements of its manifold X, the points x_i constitute the ordered set. Yet there are other kinds of fundamental sets which also permit us to construct the consistent geometries on them. In this paper we shall investigate Posets and the fuzzy ordered sets (Fosets); in this case, their elements can be incomparable or weakly ordered relative to each other [7, 8]. Based on Foset structure, novel (commutative) fuzzy geometry was constructed during the 1960s and 1970s, it will be studied here as the temptative spacetime and phase-space geometry [9–11].

In classical mechanics in the one-dimensional space $X = R^1$ the Newtonian particle is defined as 'material' point $x^m(t)$, ordered on R^1 set, i.e. relative to all its elements $\{x_a\}$. Analogously to it, in our approach the massive particle corresponds to the fuzzy point $b_m(t)$ in fuzzy space manifold C^F. Due to its weak ordering, such a particle possesses the principal uncertainty of x coordinate, i.e. it is smeared in R^1 space with an arbitrary dispersion σ_x [2]. In such a theory the quantization by itself can be defined as the transition from the ordered phase space to the fuzzy one, i.e. the quantum properties of particles and fields are induced directly by fuzzy geometry of their phase space and not postulated separately from its geometric structure. In this paper as the simple example of such a transition the quantization of the nonrelativistic particle will be regarded; it will be shown that fuzzy geometry induces the particle's dynamics which is equivalent to Schrödinger QM dynamics.

Earlier it was shown that the fuzzy observables are the natural generalization of QM observables [12]. In the last few years it was also found that some fuzzy set features are also appropriate for quantum logics formalism ([13] and references therein). It is worth mentioning here the extensive studies of quantization on noncommutative fuzzy spaces, both finite (sphere, tori) and infinite ones; this formalism exploits the methods of modern algebraic geometry [14, 15]. In these terms, we study the commutative fuzzy spaces; it turns out that such an alternative depart from Euclidean ansatz also results into the simple and consistent quantization. In section 2, we shall study the structure of particle's states induced by fuzzy geometry and discuss semiqualitatively the main features of their evolution. Based on these considerations, in section 3 the evolution equations for a free particle and the particle in the external field will be derived; it will be shown that they are equivalent to Schrödinger QM formalism. The first results of our theory were published in [2].

2. Fuzzy geometry and fuzzy states

Now we shall consider the connection between fuzzy geometry and fuzzy mechanics (FM), analogously to the connection between Euclidean geometry and classical mechanics. We shall not review here fundamentals of fuzzy geometry, which can be found elsewhere [9, 10], and consider only the simple examples important for our formalism. Recall that for elements of the partially ordered set (Poset) $D = \{d_i\}$, besides the standard ordering relation between its elements $d_k \leqslant d_l$ (or vice versa), the incomparability relation $d_k \wr d_l$ is also permitted. If it is fulfilled, then both $d_k \leqslant d_l$ and $d_l \leqslant d_k$ propositions are false. To illustrate its meaning, consider Poset $D^T = A \cup B$, which includes the subset of 'incomparable' elements $B = \{b_j\}$, and the ordered subset $A = \{a_i\}$. In A the elements' indices grow correspondingly to their ordering, so that $\forall\, i, a_i \leqslant a_{i+1}$. Any b_j is incomparable at least to one a_i. Consider some interval $\{a_l, a_{l+n}\}$, i.e. D^T subset for which $\forall\, a_i, b_j, a_l \leqslant a_i, b_j \leqslant a_{l+n}$; $n \geqslant 2$. Let us suppose that some $b_j \in \{a_l, a_{l+n}\}$ and b_j is incomparable with all $\{a_l, a_{l+n}\}$ internal elements: $b_j \wr a_i$, iff $l + 1 \leqslant i \leqslant l + n - 1$. In this case, b_j in some sense is 'smeared' inside $\{a_l, a_{l+n}\}$ interval, i.e. this is the discrete analogue of space coordinate uncertainty, if to regard A as the analogue of coordinate axis.

Fuzzy relations can be considered as the generalization of regarded incomparability relations which introduce the positive measure of incomparability w. To define it, let us put in correspondence to each b_j, a_i pair of D^T set the weight $w_i^j \geqslant 0$ with the norm $\sum_i w_i^j = 1$. The simplest example is the homogeneous incomparability: $w_i^j = \frac{1}{n}$ for regarded $a_i \in [a_l, a_{l+n}]$ interval; $w_i^j = 0$ outside it. It can be interpreted as b_j homogeneous smearing inside $[a_l, a_{l+n}]$. If w is defined for all a_i, b_j pairs in D^T, then D^T is Foset D^F, and b_j are the fuzzy points [9]. The continuous one-dimensional Foset C^F is defined analogously; $C^F = B \cup X$ where B is the same as above, X is the continuous ordered subset. If the constant metrics is defined on X, then it is equivalent to R^1 axis of real numbers. Fuzzy relations between b_j, x_a are described by the continuous distribution $w^j(x_a) \geqslant 0$ with the norm $\int w^j \,\mathrm{d}x = 1$; in this case, C^F is called the fuzzy space. Note that in fuzzy geometry $w^j(x)$ does not have any probabilistic (stochastic) meaning but only the algebraic one [10].

The particle state in classical mechanics corresponds to the ordered point $\{\vec{r}(t), \vec{p}(t)\}$ in the six-dimensional Euclidean phase space $R^3 * R^3$. In FM the nonrelativistic particle m in one-dimension is described as the fuzzy point $b(t)$ in C^F manifold (its modification for three-dimensions will be regarded in the final chapter). This means that the particle is characterized by the positive density $w(x,t)$ in the one-dimensional space R^1 with constant norm: $\int w\,\mathrm{d}x = 1$. It does not exclude, naturally, the existence of other m degrees of freedom on which its evolution can depend. In the nonrelativistic case, the time t is taken to be the real parameter on the T-axis; the particle's evolution in FM is assumed to be reversible. FM supposedly possesses the invariance relative to the space and time shifts, also it is invariant under the space and time reflections.

We suppose that m properties in an arbitrary reference frame (RF) are described by a fuzzy state $|g(t)\}$; the used notation stresses its difference from the Dirac quantum state $|\psi\rangle$. In the regarded approach, it is natural to start by assuming that, besides $w(x)$, other g independent components are the real functions of one or more coordinates, i.e. are the fields:

$$\left\{g_i^1(x)\right\}, \qquad i = 1, l_1, \qquad \left\{g_j^2(x, x')\right\}, \qquad j = 1, l_2, \ldots, \text{etc},$$

where $g_1^1(x) = w(x)$. Here and below t is omitted wherever the dependence on it is obvious. The structure of $|g\}$ states set M_s is not postulated; in particular, it is not assumed to be the linear space of any kind *a priori*. In this framework, g evolution is supposedly described by the first order on a time differential equation; it is expressed by the 'fuzzy' map $|g(t)\} = \hat{U}(t)|g_0\}$ which will be studied in the next section. We shall construct FM as the minimal theory, i.e. at every stage of its formulation it assumed that the number of $|g\}$ degrees of freedom and theory free parameters is as minimal as necessary for the theory consistency. In general, the FM formalism will be based mainly on geometric premises; in this vein, it is to some extent analogous to formalism of general relativity. In this section, we shall try to find the temptative $|g\}$ structure and some of its evolution properties from simple arguments prompted by fuzzy geometry.

Analogously to QM, besides the pure fuzzy states, we shall use for the comparison also the mixed fuzzy states g^{mix} which are the probabilistic ensembles of several fuzzy states $|g_i\}$ presented with probabilities P_i [16]. It also supposed that an arbitrary m initial state $|g_0\}$ can be prepared by some experimental procedure. To study FM dynamics, it is sensible to start from the simplest m initial states $|g_0\}$, which are point-like with $w_0(x) \sim w_1^0\delta(x - x_1)$ or some combinations of them. In the minimal FM ansatz for point-like m initial state (source) its free evolution results in m density:

$$w(x,t) = \Gamma_w(x - x_1, t)w_1^0$$

where Γ_w is the w propagator. The simple example of such evolution gives the classical diffusion [17]. In one-dimension for a point-like source in $x = 0$ one obtains

$$\Gamma_D(x,t) = \frac{1}{2\kappa\sqrt{\pi t}} \exp^{-\frac{x^2}{4\kappa^2 t}} \tag{1}$$

where κ is the diffusion constant. In this section $\Gamma_w = \Gamma_D$ will be used in the toy model illustrating the novel features of FM evolution; the detailed description of this model can be found in [1]. Its use is instructive, because the main FM distinction from classical mechanics lays in the correlations between g components at different x points and not in the evolution of a point-like state. As will be shown below, in FM the exact effective Γ_w solutions do not differ principally from Γ_D.

This difference between FM and classical mechanics can be illustrated by the effect of m sources smearing (SS) or indistinguishability which is the direct analogue of quantum interference [18]. In its essence, depending on the fuzzy or classical structure of initial m state (source), the $w(x,t)$ form can differ dramatically, whereas $\bar{x}$ will be practically the same. To demonstrate this, let us consider one-dimensional analogue of notorious two-slit experiment (TSE) of QM [16, 19]. We shall consider the system of $n_s = 2$ point-like m sources (bins) with $Dx_{1,2}$ width cited in $x_{1,2}$. Consider first the probabilistic mixture g_0^{mix} of $g_{1,2}^0$ states localized in $Dx_{1,2}$, respectively. In this case, the weight $w_i^0 = P_i$ where P_i is the probability for m to be in Dx_i; the density of m sources is

$$w^0(x) = \sum^{n_s} w_i^0 \delta(x - x_i) \tag{2}$$

over m ensemble (we consider the mixtures in which $w_{1,2}^0$ are the same or do not differ much). In each individual event m is emitted definitely by Dx_1 or Dx_2 at t_0, therefore the g_0^{mix} algebraic structure is described by the following proposition:

$$LP^{\text{mix}} := \qquad m \in Dx_1.or.m \in Dx_2.$$

Consequently, the resulting m distribution over this ensemble at any $t > t_0$ will be the additive sum:

$$w_{\text{mix}}(x,t) = w_1(x,t) + w_2(x,t) = \sum w_i^0 \Gamma_w(x - x_i, t).$$

For SS illustration the most interesting is the case when $w_{1,2}(x,t)$ intersect largely, i.e. for $L_x = |x_1 - x_2|$ it should be $L_x \leqslant \sigma_x(t)$ where $\sigma_x(t)$ is $w_{1,2}$ dispersion. For our toy model it holds if $L_x \leqslant \kappa t^{\frac{1}{2}}$. The rate of w_1, w_2 overlap can be estimated as

$$R_w = 2 \int \sqrt{w_1 w_2}\, \mathrm{d}x$$

and it should not be much less than 1.

Now consider the pure fuzzy state $|g_0\}$ for which m coexists simultaneously in both bins Dx_i with the same weights w_i^0; more precisely, g_0 is supposed to be the superposition of g_i^0 states of the regarded mixed ensemble (exact FM definition of state's superposition will be given below). At this stage it is enough to admit that for such (pure) m state $|g_0\}$ the following proposition describes m source structure:

$$LP^s := m \wr Dx_1.\text{and}.m \wr Dx_2$$

where $m \wr Dx_i$ means that $m \wr x_a; \forall x_a \in Dx_i$. In this case, LP^{mix} and LP^s are incompatible:

$$LP^{\text{mix}}.\text{and}.LP^s = \emptyset$$

The incompatibility of LP^s, LP^{mix} indicates that the signal of fuzzy source S cannot be decomposed into the sum of signals from local sources $Dx_{1,2}$. For such source's system from $w(x) = w_{\text{mix}}(x)$ follows $LP^{\text{mix}} =$ .true. with definiteness. Hence, if the resulting distribution w_s is to decompose as

$$w_s(x) = w_p(x) + k_w w_{\text{mix}}(x)$$

where $w_p \geqslant 0$ is arbitrary, then it follows that $k_w = 0$, i.e. any w_{mix} content in w_s is excluded. If w_s, w_{mix} supports in X mainly coincide, such a k_w value is possible only if w_s oscillates around w_{mix} and at one or more points x_j where $w_{\text{mix}}(x_j) \neq 0$ it gives $w_s(x_j) = 0$. Simply speaking, such a picture describes the interference patterns similar to those observed for QM superposition. w_s can be decomposed as

$$w_s(x) = w_1(x) + w_2(x) + w_n(x)$$

where w_n is the nonlinear term. For our toy model it can be expressed as

$$w_n(x) \sim 2\cos\left[r_D\left(x - \frac{x_1 + x_2}{2}\right)\right][w_1(x)w_2(x)]^{\frac{1}{2}}$$

where r_D is arbitrary but $r_D \gg \kappa\sqrt{t}$ [1]. One should also define the SS measure, i.e. the criteria of signals separation R_{ss} for the evaluation of the smearing rate; depending on it R_{ss} can vary from 0 for g^{mix} to 1 for the fuzzy state with maximal SS. The general R_{ss} ansatz is quite complicated [1], but R_{ss} will be used in our formalism only in the asymptotic limits when $R_{ss} \to 0$ or 1. In FM framework, the m free evolution SS effect should respond to the maximal R_{ss} value, because in FM no information about m path from the source $|g_0\}$ at all exists. Otherwise, it would mean that the additional information about m source (path) is produced stochastically during m evolution, but it is impossible because of FM reversibility.

A similar SS effect should be expected for complete FM formalism; hence it is instructive to exploit whether fuzzy geometry prompts some indications for the SS geometric scale characterized by $\sigma_x(t)$. By itself, fuzzy geometry does not contain any length parameters which can be put in correspondence to σ_x. Actually, the fuzzy point b_j described by $w_j(x)$ possesses the obvious scaling properties for w_j dispersion. From that it is quite natural to expect that in the one-dimensional FM the influence of source g_0^i on the state $|g(t)\}$ at point x is independent of $|x - x_i|$. Hence minimal FM should also show the scaling behaviour, which permit us to omit any length parameters settling $\sigma_x(t) \to \infty$; $\forall\, t$. In the relativistic theory, the dispersion $\sigma_x(t)$ is restricted by the maximal velocity c, so that $\sigma_x \leqslant ct$. In the nonrelativistic case, nothing forbids us from choosing the FM ansatz for the point-like source x_i such that at $x \to \pm\infty$, $\lim w(x - x_i, t) \neq 0$ (or the limits do not exist) [20]. This $w(x - x_i, t)$ property is called the x-limit condition; in our toy model it is fulfilled only for $t \to \infty$. Then $w(x)$ should be the Schwartz distribution (generalized function) [17]. Such $|g\}$ evolution first seems quite exotic; recall yet that in QM the point-like initial state in one dimension evolves analogously [19].

Consider now the system of $n_s = 2$ sources with particular $x_{1,2}$, $w^0_{1,2}$; each state $g^0_{1,2}$ evolves into $w_{1,2}(x, t)$ which satisfies the x-limit condition. If $|g_0\}$ is their superposition then the resulting $w_s(x, t)$ should also satisfy this. $\bar{x}(t)$ and higher x-moments are undefined for such $w_s(x)$ and in this case only the $w_s(x)$ form can depend on FM dynamics. In FM this $w_s(x)$ should also correspond to the maximal SS, i.e. $R_{ss} \to 1$. Then $w'_s(x, t) = w_s(x + a_x, t)$ also corresponds to it for an arbitrary a_x, because R_{ss} depends on the w_s form only. If R_{ss} maximality is the only condition of $|g(t)\}$ consistency, then w'_s can also be the solution for some g_0 state which is the superposition of g^0_i. This conclusion is especially obvious if $w_i(x, t)$ are practically independent of x; in our toy model it occurs for $t \to \infty$. Hence the resulting a_x value should be defined by the initial $|g_0\}$ state; these considerations evidence that besides $w(x)$, $|g\}$ includes at least one more degree of freedom. Since a_x depends on $|g\}$ in both x_1 and x_2, it is sensible to assume that it can be represented as the correlation field $g^2_i(x_1, x_2)$ introduced above. In minimal FM for the arbitrary state $|g\}$ it can be an arbitrary real function of two variables $g^2_1 = K^f(x, x')$ which is continuous or has a finite number of breaking points. Consequently, for the $n_s = 2$ system a_x is some function: $a_x = f^f[K^f(x_1, x_2)]$. If we choose the gauge: $\forall\, x_b$, $K^f(x_b, x_b) = 0$, then regarding the fixed x_c as the parameter we obtain

$$K^f(x_d, x_c) = \int_{x_c}^{x_d} \frac{\partial K^f(\xi, x_c)}{\partial \xi}\, \mathrm{d}\xi$$

and from that follows

$$K^f(x_d, x_e) = K^f(x_d, x_c) - K^f(x_e, x_c).$$

Therefore K^f is, in fact, the function of one observable: $\lambda(x) = K^f(x, x_c)$. Hence g can be treated as the local field $E^g(x) = \{w(x), \lambda(x)\}$. One can transform it into the symmetric $|g\}$

representation by the complex function $g(x) = w^{\frac{1}{2}}(x)\exp \mathrm{i}c_\lambda\lambda(x)$ where the c_λ parameter will be calculated below. In this case, w/g zero-equivalence holds: $w(x) = 0 \Leftrightarrow g(x) = 0$ and the same is true for w, g limits at $x \to \infty$. If the x-limit condition is fulfilled for $w(x, t)$, then it is also true for $g(x, t)$ which should also be Schwartz distribution. Generally, one should be careful with the interpretation of $w(x, t)$ distributions as the measurable distributions of physical parameters, yet in the discussion of QM foundations it is admissible to regard them as the standard, normalized functions, as was demonstrated in [19]. This problem will be reconsidered in detail below.

3. Particle's evolution in fuzzy dynamics

From the previous discussion we deduced that in minimal FM the state of particle m in one-dimensional space X is described by the normalized complex function $g(x, t)$; for free m evolution from the point-like source it satisfies the x-limit condition. In general, the $g(x, t)$ reversible evolution is described by the parameter-dependent unitary operator $\hat{U}(t)$, so that $g(t) = \hat{U}(t)g_0$. It possesses the properties of group element:

$$U(t_1 + t_2) = U(t_1)U(t_2), \qquad \forall t_{1,2}.$$

Therefore m free evolution can be expressed as $\hat{U}(t) = \mathrm{e}^{-\mathrm{i}\hat{H}_0 t}$ where $\hat{H}_0$ is an arbitrary constant operator [18]. It is not supposed to be linear beforehand, but we start from the consideration of linear H_0; the obtained results will help us to analyse the nonlinear case. The free g evolution is invariant relative to X shifts performed by the operator $\hat{V}(a) = \exp\left(a\frac{\partial}{\partial x}\right)$. Because of it, $\hat{U}(t)$ should commute with $\hat{V}(a)$ for the arbitrary a. It is equivalent to the relation $\left[\hat{H}_0, \frac{\partial}{\partial x}\right] = 0$, from which it follows that $\hat{H}_0$ in p representation is an arbitrary function of p: $H_0 = F_0(p)$.

Consider now the initial point-like state $|g_0\}$ inducing m density $w^0 = \delta(x - x_0)$; we put in correspondence to it the unnormalized function $g_0(x) = \exp(\mathrm{i}\alpha_0)\delta(x - x_0)$ where α_0 is an arbitrary real number. The proper g_0 normalization will be regarded below, at this stage it will introduce the unnecessary complications but would not change g ansatz in an essential way. Then from the $\delta(x - x_0)$ Fourier transform $\varphi_\delta(p) = \exp(\mathrm{i}px_0)$ it follows that the g Fourier transform is equal to

$$\varphi(p, t) = U(t)\,\mathrm{e}^{\mathrm{i}\alpha_0}\varphi_\delta = \mathrm{e}^{-\mathrm{i}F_0(p)(t-t_0)+\mathrm{i}px_0+\mathrm{i}\alpha_0}$$

below $x_0 = 0, t_0 = 0$ is assumed. The transition $\delta(x) \to g(x, t)$ develops continuously without breaking points if $g(x, t_j)$ constitutes δ-sequence, i.e. $g(x, t_j) \to \delta(x)$ for any sequence $\{t_j\} \to +0$ [17]. This condition is fulfilled only if $g(x, t)$ has $t = 0$ pole, so that $g(x, t)$ can be decomposed as $g = g_s g_a$ where for the substitution $z = \frac{x}{f(t)}$ one obtains

$$g_s(z, t) = \frac{1}{f(t)}\,\mathrm{e}^{\mathrm{i}\gamma(z)},$$

with an arbitrary, complex γ; $f(t) \to 0$ at $t \to +0$. g_a is an arbitrary, nonsingular function with $g_a(x, t) \to 1$ at $t \to +0$, so it can be neglected in this limit. If in that case the asymptotic relation

$$\int_{-\infty}^{\infty} g(z, t) f(t)\,\mathrm{d}z \to 1$$

is fulfilled, then under these conditions $g(x, t) \to \delta(x)$ at $t \to +0$. After z substitution g Fourier transform φ can be alternatively represented as

$$\varphi'(p, t) = c_0 \int_{-\infty}^{\infty} \mathrm{d}z\,\mathrm{e}^{\mathrm{i}\gamma(z)+\mathrm{i}zpf(t)} = \exp^{-\mathrm{i}\Lambda[pf(t)]}. \tag{3}$$

From the equivalence of $\varphi(p,t)$ and $\varphi'(p,t)$ one obtains the equation

$$\varphi(p,t) = \mathrm{e}^{-\mathrm{i}F_0(p)t} = \exp^{-\mathrm{i}\Lambda[pf(t)]+\alpha_e} \tag{4}$$

from which follows $F_0(p) = \frac{p^s}{2m_0}$, $f(t) = d_r t^r, \alpha_e = 0$, with $rs = 1$, where m_0, d_r are arbitrary parameters. If $H_0 = F_0(p)$ is regarded as m free Hamiltonian, then from its symmetry properties and the energy positivity it follows that $m_0 > 0$ and the consistent s values are only the natural even numbers.

Let us consider first the case $s = 2$; it follows that the free Hamiltonian is $H_0 = \frac{p^2}{2m_0}$. For the point-like state $g_0(x) = \mathrm{e}^{\mathrm{i}\alpha_0}\delta(x - x_0)$ one obtains

$$\varphi(p,t) = \mathrm{e}^{-\frac{\mathrm{i}p^2 t}{2m_0}}$$

which in x-representation results in

$$g(x,t) = G(x - x_0, t)\,\mathrm{e}^{\mathrm{i}\alpha_0} = \sqrt{\frac{m_0}{-\mathrm{i}2\pi t}}\,\mathrm{e}^{\frac{\mathrm{i}m_0(x-x_0)^2}{2t}+\mathrm{i}\alpha_0}, \tag{5}$$

hence for a positive m_0 value G coincides with the QM free propagator for a particle with mass m_0 up to arbitrary constant [19]. If we assume that G is a FM free m propagator, then an arbitrary normalized function describing the initial state $g_0(x) = \sqrt{w_0(x)}\,\mathrm{e}^{\mathrm{i}\theta(x)}$ will evolve as

$$g(x',t) = \int G(x' - x, t) g_0(x)\,\mathrm{d}x = \sqrt{\frac{m_0}{-\mathrm{i}2\pi t}} \int \mathrm{e}^{\frac{\mathrm{i}m_0(x'-x)^2}{2t}} g_0(x)\,\mathrm{d}x \tag{6}$$

which coincides with the free g_0 evolution in QM formalism [19]. For such an evolution ansatz one finds that the integral form $\int |g(x,t)|^2\,\mathrm{d}x$ is time independent and equal to 1; in this case, $w = |g|^2$ satisfies the m flow conservation equation. Note that for $s = 2$, $g(x,t) \neq 0$ at $x \to \pm\infty$, i.e. satisfies the x-limit condition, as minimal FM ansatz assumes. Yet it is violated for free Hamiltonian with $s \geqslant 4$; in this case, the $g(x,t)$ asymptotic can be calculated [21] at $x \to \pm\infty$:

$$g(x,t) \simeq \frac{c_g}{t^{\frac{1}{s}}} \left(\frac{t^{\frac{1}{s}}}{x}\right)^{\frac{s-2}{2(s-1)}} \exp^{\mathrm{i}\frac{s-1}{s} m^{\frac{1}{2(s-1)}} t^{-\frac{1}{2(s-1)}} x^{\frac{s}{2(s-1)}}}$$

with c_g an arbitrary constant. In particular, for $s = 4$, $|g| \sim \frac{1}{|x|^{\frac{1}{3}}}$. Therefore $g \to 0$ at $x \pm \infty$, so it contradicts the x-limit condition; hence this crucial assumption of minimal FM is violated for $s \geqslant 4$.

Let us consider now the general case of FM free evolution, which does not demand, in principle, that the $\hat{U}(t)$ operator should be linear. However, m evolution is supposed to be reversible, so $\hat{U}(t)$ must be unitary. The thorough investigations of nonlinear Schrödinger-type operators have shown that such physically nontrivial operators are nonunitary [22]. In accordance with it, we shall demonstrate that the unitary free m evolution cannot be induced by nonlinear Hamiltonian $\hat{H}_0$. Here we only sketch the proof leaving some mathematical details for the future study. Consider an arbitrary m normalized state $|g\}$ in p-representation:

$$\varphi(p,t) = [w_p(p,t)]^{\frac{1}{2}}\,\mathrm{e}^{\mathrm{i}\beta(p,t)}$$

where β is real; it obeys the equation

$$-\mathrm{i}\frac{\partial\varphi}{\partial t} = \hat{H}_0\varphi.$$

Since free H_0 is invariant relative to the space shifts, then $\langle p^n\rangle$ are constant $\forall\, n$; from that it follows that $\frac{\partial w_p}{\partial t} = 0$. It transforms the latter equation into

$$\frac{\partial\beta}{\partial t}\varphi = \hat{H}_0\varphi, \tag{7}$$

i.e. H_0 action results in multiplication of φ on some function $F(\varphi, p)$. H_0 is the constant operator, hence F cannot openly depend on t. The obvious solution is $\hat{H}_0 = F(p)$, and it just corresponds to the regarded linear ansatz with $F = F_0(p)$. The simple analysis shows that no other physically interesting $\hat{H}_0$ solutions exist.

Now the normalization of $g(x, t)$ states from point-like m sources will be considered; it is also valid for QM formalism where this aspect is often missed [19]. This problem is quite trivial, so in place of universal derivation we shall regard it for a particular δ-sequence. Plainly, the state of the point-like source $g_0(x)$ should be the limit of physical normalized states of very small width. Namely, it can be the sequence of initial states:

$$\eta_\sigma = \frac{\mathrm{e}^{-\frac{x^2}{2\sigma_x^2}}}{\pi^{\frac{1}{4}}\sigma^{\frac{1}{2}}}$$

for $\sigma \to 0$; the resulting function $\delta^\eta(x) = \lim \eta_\sigma(x)$ called the squire root of $\delta(x)$. η_σ density $w_\sigma(x)$ has the norm 1 and the limit $\delta(x)$, as expected for the state of point-like source. Hence it seems consistent to choose $g_0(x) = \mathrm{e}^{\mathrm{i}\alpha_0}\delta^\eta(x)$ as the point-like state in FM (and QM also). Its Fourier transform

$$\varphi_\eta(p, t) = \lim_{\sigma \to 0} \frac{\sigma^{\frac{1}{2}}}{(2\pi)^{\frac{1}{4}}} \mathrm{e}^{\mathrm{i}\alpha_0 - \mathrm{i}\frac{p^2 t}{2m_0} - 2\sigma^2 p^2}$$

also has the norm 1 at any t. φ describes the normalized constant distribution of m density on the p axis. If to substitute such g_0 into (6), the resulting $g(x', t)$ will have norm 1 at any t. It stresses that the propagator G is not the physical state of particle m. However, all FM results obtained above do not depend on this renormalization and stay unchanged, because such renormalization is, in fact, the multiplication of $g_0(x)$ and $g(x, t)$ on the infinitesimal constant.

Now Hamilton formalism for FM can be formulated consistently. In our theory m momentum is the operator $\hat{p} = -\mathrm{i}\frac{\partial}{\partial x}$ [16] in x representation and the free Hamiltonian $\hat{H}_0 = \frac{\hat{p}^2}{2m_0}$. In FM the natural $U(t)$ generalization for the m potential interactions $V_m(x)$ is $\hat{H} = \hat{H}_0 + V_m(x)$. From obtained relations it results in Schrödinger equation for g; the general path integral ansatz for g can be obtained by means of Lagrangian $\mathcal{L}$ derived from $\hat{H}$ for the given $V_m(x)$ [19]. Any normalized function $g(x)$ admits the orthogonal decomposition on $|x_a\rangle = \delta(x - x_a)$, and the $|x_a\rangle$ set constitutes the complete system [18]. Therefore $|g\}$ set M_s is equivalent to the complex rigged Hilbert space $\mathcal{H}$ with the scalar product $g_1 * g_2 = \int g_1^* g_2 \, \mathrm{d}x$. Consequently, our theory does not need superposition principle as the independent axiom, it follows from other FM axioms. In FM x is an m observable and it is sensible to suppose that $\hat{p}$ and any Hermitian operator function $\hat{Q}(x, p)$ are also m observable. For any such Q there is the corresponding complete system of orthogonal eigenvectors $|q_a\rangle$ in $\mathcal{H}$. It is allowed to assume that for FM measurements of observables QM reduction (projection) postulate for an arbitrary observable Q can be incorporated in FM copiously [16]. Generalization of FM formalism on three dimensions is straightforward, the only novelty is that $|g\}$ correlation between two points $\vec{r}_{1,2}$ defined in the previous section:

$$K^f(\vec{r}_1, \vec{r}_2) = \int_l \frac{\partial K^f(\vec{r}, \vec{r}_2)}{\partial \vec{r}} \, \mathrm{d}\vec{l}$$

is supposed to be independent (up to $2\pi n$) of the path l over which it is calculated. In this case, the $|g\}$ quantum phase $\alpha(\vec{r})$ is defined unambiguously.

It turns out that the obtained $\hat{U}(t)$ ansatz coincides with the QM Schrödinger evolution operator for free m evolution. The analogous results for QM are obtained in the theory of the irreducible representations, but in that case they are based on more complicated axiomatic;

in particular, it includes the axiom of Galilean invariance [16]. In distinction, FM does not assume Galilean invariance of g states in different RFs but only the invariance relative to the space and time shifts. It acknowledged in quantum physics that the classical massive objects, including physical RFs, can be regarded as the quantum objects in the limit $m_0 \to \infty$ [16]. If such an approach is correct in the FM framework also, then regarding m with $m_0 \to \infty$ as RF, Galilean transformations can be derived from the obtained FM ansatz for H_0. Of course, this hypothesis needs further investigation, but in this approach it seems consistent. Note that Planck constant $\hbar = 1$ in our FM ansatz, analogously to relativistic QM alike; in the FM framework, it only relates x, p scales in our formalism and does not have any other meaning [19]. The proposed FM considers the nonrelativistic particle for which x is the fuzzy coordinate, yet from the symmetry of phase space one can choose any observable Q as the fundamental fuzzy coordinate and from this assumption to reconstruct FM formalism. It can be especially important in the relativistic case where x cannot be the proper observable [16]. In addition, the linearity of state evolution becomes the important criterion for the choice of consistent ansatz. For a massive particle the minimal solution is 4-spinor, i.e. its evolution is described by the Dirac equation for spin-$\frac{1}{2}$ [16].

In conclusion, we have shown that the quantization of elementary systems can be derived directly from axiomatic of sets theory and topology together with the natural assumptions about systems evolution. For example, in commutative fuzzy space C^F the resulting x, p noncommutativity is induced by minimal properties of m evolution, first of all, its invariance relative to time shifts. Note that FM quantization does not need the corresponding classical system as the starting point [15]. Copenhagen QM interpretation claims that QM cannot be formulated consistently without the preliminarily postulated classical notions, but it seems that FM formalism is, at least, essentially less connected with them. Thereon, it allows us to suppose that the quantization phenomenon has its roots in foundations of mathematic and logics [16]. The FM approach, in principle, can be extended on quite different physical systems. Here we considered only the fuzzy phase space of single particle, but such phase space of any kind can be constructed. In particular, it can be the Fock space for the secondary quantization, in this case; the occupation numbers for particle's states $N_c(\vec{p})$ can be regarded as the fuzzy values. Yet the main target of FM, as well as of other studies of fuzzy spaces, is the construction of nonlocal QFT (or other more general theory) [15]. In this vein, FM provides the interesting opportunities, being generically nonlocal theory which, in the same time, is Lorentz covariant. It can help to reformulate some old methods of nonlocal QFT, in particular, the formalism of nonlocal ghost fields [23]. If such theoretical development will be successful, it can change in an essential way the description of some fundamental effects, in particular, of vacuum fluctuations and particle's self-energy [5, 23].

References

[1] Mayburov S 2002 *Proc. Quantum Foundations Conf.* (Växjö: Växjiö University Press) p 232 (*Preprint* hep-th/0210113)

[2] Mayburov S 2004 *Proc. Steklov Math. Inst.* **245** 154

[3] Aharonov Y and Kaufherr T 1984 *Phys. Rev.* D **30** 368

[4] Doplicher S, Fredenhagen K and Roberts K 1995 *Commun. Math. Phys.* **172** 187

[5] Isham C 1994 *Canonical Gravity: from Classical to Quantum (Lecture Notes in Phys.* vol 433*)* ed J Ehlers and H Friedrich (Berlin: Springer) p 11

[6] Vladimirov V S and Volovich I V 1989 *Commun. Math. Phys.* **659** 123

[7] Zadeh L 1965 *Inf. Control* **8** 338

[8] Bandemer H and Gottwald S 1993 *Einführung in Fuzzy-Methoden* (Berlin: Akademic Verlag) p 45

[9] Zeeman E C 1961 *Topology of 3-Manifolds* (Englewood cliff, NJ: Prentice-Hall) p 240

[10] Dodson C T J 1975 *J. Lond. Math. Soc.* **2** 465

[11] Poston T 1971 *Manifold* **10** 25
[12] Ali T and Emch G 1974 *J. Math. Phys* **15** 176
[13] Pykaz J 2000 *Found. Phys.* **30** 1503
[14] Madore J 1992 *Class. Quantum Grav.* **9** 69
[15] Balachandran A P, Kürkcüuoğlu S and Vaidia S 2005 *Lect. fuzzy and fuzzy SUSY phys. Preprint* hep-th/051114
[16] Jauch J M 1968 *Foundations of Quantum Mechanics* (Reading, MA: Addison-Wesley)
[17] Vladimirov V S 1971 *Equation of Mathematical Physics* (Moscow: Nauka)
[18] Berezin F and Shubin E 1985 *Schrödinger Equation* (Moscow: Nauka)
[19] Feynman R and Hibbs A 1965 *Quantum Mechanics and Path Integrals* (New York: McGraw-Hill)
[20] Schwartz L 1961 *Méthods Mathématiques pour les Sciences Physique* (Paris: Hermann)
[21] Fedoriuk M 1987 *Asymptotics: Integrals and Rows* (Moscow: Nauka)
[22] Doebner H D and Goldin G A 1996 *Phys. Rev.* A **54** 3764
[23] Blokhintsev D I 1982 *Space-Time in Microworld* (Moscow: Nauka)